Holographic entanglement in vacuum

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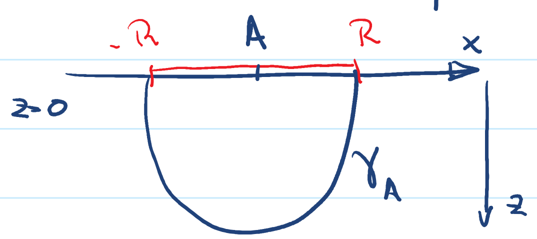
Let's begin w/ simple calculations:

AdS$_3$ vacuum: $$ds^2 = \frac{dz^2 - dt^2 + dx^2}{z^2}$$
Poincaré

Consider an interval of length $2R$ at the boundary



We must find a curve γ_A that minimizes the length between its endpoints at $z=0$, $x=-R, R$. This is a geodesic.

Parametrizing the curve as $z(x)$

Length element is
$$d\ell = \sqrt{\frac{z'(x)^2 + 1}{z(x)^2}}\, dx$$

Then minimize
$$S = \frac{1}{4G}\int dx \frac{\sqrt{z'(x)^2+1}}{z(x)}$$

and impose that $\quad z(\pm R) = 0$

It's easy to find that the minimal curves are semi-circles

$$z = \sqrt{R^2 - x^2}$$

$$\frac{\sqrt{z'(x)^2 + 1}}{z(x)} = \frac{R}{z^2}$$

$$\int_{-R}^{R} dx \, \frac{\sqrt{z'^2+1}}{z} = \int_{-R}^{R} dx \, \frac{R}{z^2(x)} = 2R \int_0^{z_{max} = R} \frac{dz}{z \, x(z)}$$

This is log-divergent near $z = 0$

Introducing a cutoff at $z = \varepsilon$, the integral can be easily evaluated:

$$S = \frac{1}{2G} \log \frac{2R}{\varepsilon}$$

Using $c = \frac{3}{2G}$ we find

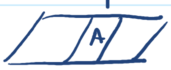
$$S = \frac{c}{3} \log \frac{2R}{\varepsilon}$$

which is exactly the same as can be derived for a general CFT_2 w/ central charge c using cft techniques

In AdS_{d+1} $\quad ds^2 = \dfrac{dz^2 - dt^2 + d\vec{x}^2_{d-1}}{z^2}$

we may consider different kinds of region A:

eg strips or spherical balls



For a spherical ball of radius R, choosing radial coordinate $r^2 = \vec{x}^2_{d-1}$, $r \leq R$, and parametrizing $z(r)$

we have to extremize

$$S = \frac{1}{4G} \int d\omega_{d-2} \, dr \, r^{d-2} \frac{\sqrt{1+z'(r)^2}}{z(r)^{d-1}}$$

The extremal surfaces are hemi-spherical caps

$$z^2 + r^2 = R^2 \qquad\qquad \sqrt{1+z'^2} = \frac{R}{z}$$

If we change $z(r) \to r(z)$, with $dr = -\frac{z\,dz}{r(z)}$

so

$$S = \frac{1}{4G} \Omega_{d-2} R \int_{\varepsilon}^{R} r(z)^{d-3} \frac{dz}{z^{d-1}} \; : \text{ diverges } \sim \frac{1}{\varepsilon^{d-2}}$$

at $z = \varepsilon$

$$\sim \frac{1}{4G} \Omega_{d-2} R^{d-2} \frac{1}{\varepsilon^{d-2}}$$

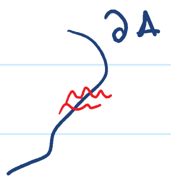
and since $\Omega_{d-2} R^{d-2} = \text{Area}(\partial A)$ then

$$\boxed{S = \frac{1}{4G} \#_d \frac{\text{Area}(\partial A)}{\varepsilon^{d-2}}}$$

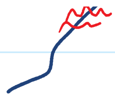
(exercise: compute $\#_d$)

This is the <u>area-law of entanglement entropy</u> of a local QFT

Entanglement in local QFT is due to short-distance correlations across the boundary ∂A



it is dominated by short-wavelength modes ([illegible]) so it is naturally



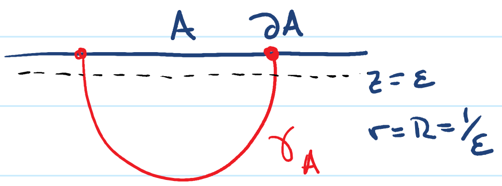
it is dominated by short-wavelength modes (locality) so it is naturally

$$S_{ent} \propto \frac{\text{Area}(\partial A)}{\varepsilon^{d-2}}$$

In the CFT this is due to the local UV structure of the quantum fields

N.B.: The area-law is valid for a quantum field theory w/ local interactions independently of whether it has a gravitational dual

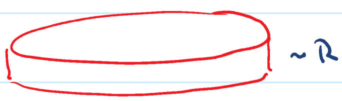
In AdS it is due to the concentration of the area of the bulk surface near the boundary $z=0$



$$\text{Area}(\gamma_A) \sim \text{Area}(\partial A)\, R^{d-2} = \frac{\text{Area}(\partial A)}{\varepsilon^{d-2}}$$

$\text{Area}(\gamma_A)$ ↓ dimension $d-1$

$\text{Area}(\partial A)$ ↓ dimension $d-2$



AdS encodes geometrically the short-distance correlations (entanglement) of a CFT in vacuum

$\boxed{\text{Entanglement of large numbers of degrees of freedom manifests itself geometrically}}$

The area-law can be seen very generally using the Fefferman-Graham expansion for a generic boundary metric:

$$ds^2 = \frac{1}{z^2}\left(dz^2 + g^{(0)}_{ab}(x)\, dx^a dx^b + O(z^2)\right)$$

Specifying a spatial region A in the boundary, with γ_A such that $\gamma_A|_{z=0} = \partial A$, we have

$$\text{Area}(\gamma_A) = \int_{z=\varepsilon}^{z_{max}} dz \, \frac{1}{z^{d-1}} \int d^{d-2}x \, \sqrt{g^{(0)}}\Big|_{\partial A} \sim \frac{\text{Area}(\partial A)}{\varepsilon^{d-2}}$$