Holographic entanglement in thermal states

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Reminder: horizon at $\underset{(t=\text{const})}{t=0}$ is a minimal surface

Consider a static black hole (can be generalized to stationary) with a non-degenerate (bifurcate) horizon, ie $T_H \neq 0$.

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + g^2(r)\,d\Sigma_n \quad w/ \quad \begin{cases} f(r_0) = 0 \\ f'(r_0) > 0 \\ g(r_0) \neq 0 \\ g'(r_0) > 0 \end{cases}$$

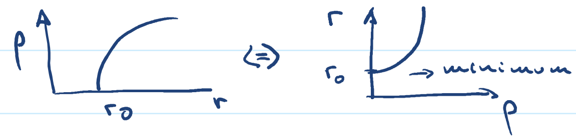
On a section at $t = \text{const}$

$$ds^2\Big|_t = \frac{dr^2}{f(r)} + g^2(r)\,d\Sigma_n$$

The surface $r = r_0$ is minimal:

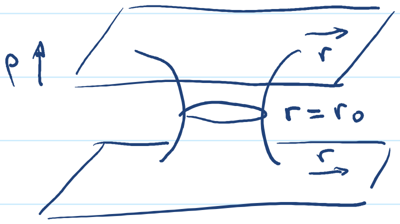
Intuitively:

proper radius $d\rho \sim \dfrac{dr}{\sqrt{f(r)}} \sim \dfrac{1}{\sqrt{f'(r_0)}} \dfrac{dr}{\sqrt{r-r_0}} \Rightarrow \rho \sim \sqrt{r-r_0}$



assume $g(r) \simeq r$ for illustration

represent Σ_n as circles



Einstein-Rosen bridge

More Technically: area element $a(r) = g(r)\,\sigma_n$

$n^r = \sqrt{f}$

$\sigma_n = \mathrm{Vol}(\Sigma_n)$

$$\partial_n a(r) = n^r\,\partial_r a(r) = \sqrt{f(r)}\,g'(r)\,\sigma_n$$

and since $f(r_0) = 0$ the area is extremal at $r = r_0$.

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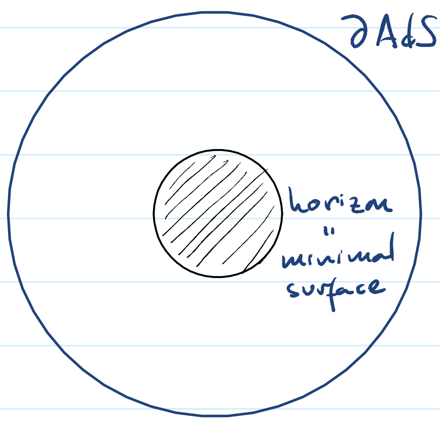
Next

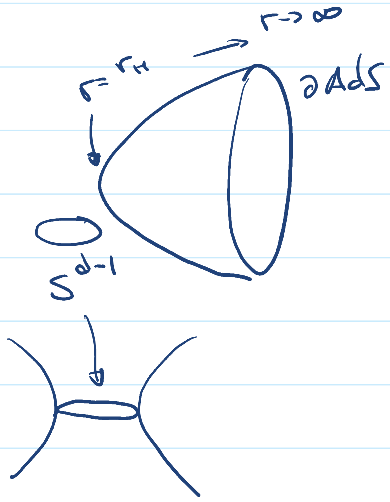
$$\frac{1}{\sigma_n}\partial_n^2 a(r) = \sqrt{f}\,\partial_r\left(\sqrt{f}\,g'\right) = \frac{f'g'}{2} + f g''$$

$$'' \quad \Big|_{r=r_0} = \frac{1}{2} f'(r_0)\, g'(r_0) > 0 \quad \text{by the assumptions above}$$

so the surface is minimal

Consider now a black hole in global AdS, in a section $t=0$





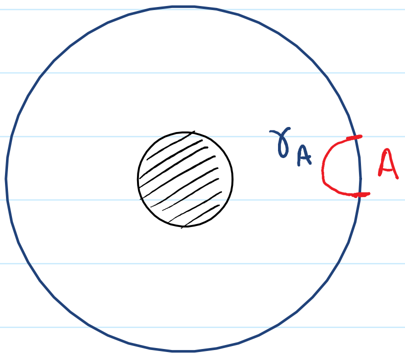
We use this black hole to study the properties of the CFT at finite temperature (above the Hawking-Page transition)

Now we want to evaluate the entanglement entropy of a region of the boundary at finite T.

For a CFT_2, these calculations are very difficult using conventional techniques (for a $CFT_{d>2}$ they're even harder). But using holography they're much easier, and correspond to finding geodesics (minimal curves) in the geometry of a black hole in AdS_3 ie the BTZ black hole (which we'll see later).

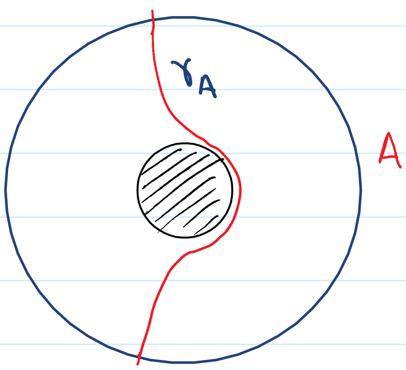
Moreover, the qualitative aspects of the calculation are easily understood, and this is what we'll see here.

When the region is small, it's much like there is no black hole:

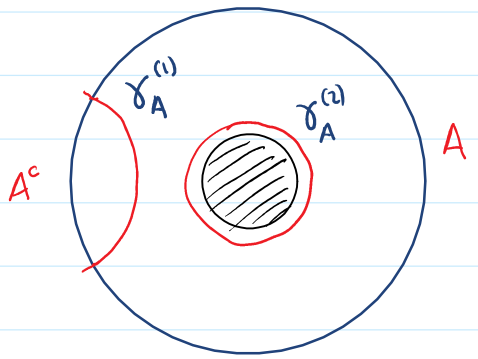


$$S_A(T) \sim S_A(\text{vacuum})$$

As A grows larger, at a size $\ell \sim \frac{1}{T_H}$ the RT surfaces begins to feel the presence of the black hole



until a moment when the minimal surface becomes disconnected (because it has lower area, or because the connected one ceases to exist, in $d \geq 3$)



$\gamma_A^{(1)} \cup \gamma_A^{(2)}$ is homologous to A

$\gamma_A^{(1)}$ homologous to A^c

$$S(A) = \frac{\text{Area}(\gamma_A^{(1)})}{4G} + \frac{\text{Area}(\gamma_A^{(2)})}{4G}$$

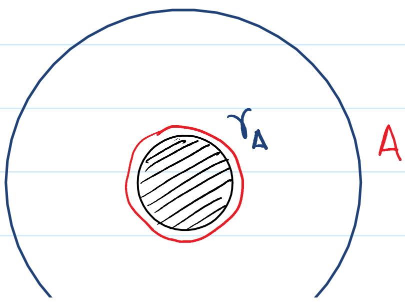
$$= S(A_c) + S_{BH}$$

Since the state is not pure, we don't have $S(A) = S(A_c)$ but rather $|S(A) - S(A_c)| = S_{BH}$

Now notice that if $A^c = \phi$, ie $A = \partial AdS$ is the entire boundary, then

$$S_{ent}(A)_{\infty} = S_{BH}$$

The entropy of the thermal state of the CFT is entanglement entropy:



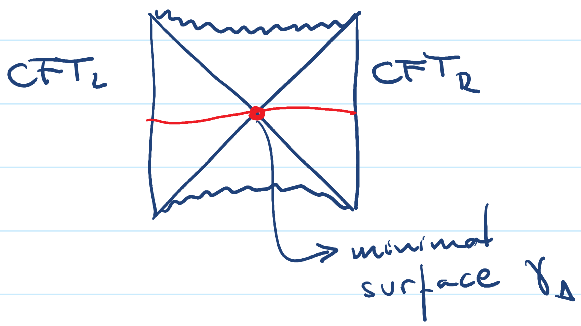
The minimal surface γ_A at the horizon is homologous to the entire

homologous to the entire boundary A, so

$$S_{ent}(A) = S_{BH}$$

and viceversa, the entropy of the bulk black hole is entanglement entropy of the CFT.

This ties in with the idea that the eternal black hole is dual to the TFD:



Thermality in CFT_R appears as the consequence of its entanglement with CFT_L

If we now consider A_R to be the right AdS bdry, then the horizon at $t=0$, γ_A, is homologous to A_R

so $S(A_R) = S_{BH} = S(A_L)$

The entanglement between CFT_R and CFT_L gives rise, holographically, to the geometry that bridges between them in the bulk.

- <u>Entanglement entropy in a thermal state on $\mathbb{R}^{1,d-1}$</u>

The bulk is the AdS black brane

$$ds^2 = r^2\left(-f(r)\,dt^2 + d\vec{x}^2_{d-1}\right) + \frac{dr^2}{r^2 f(r)} \qquad f(r) = 1 - \left(\frac{r_+}{r}\right)^d$$

$$= \frac{1}{z^2}\left(-f(z)\,dt^2 + d\vec{x}^2_{d-1} + \frac{dz^2}{f(z)}\right) \qquad f(z) = 1 - \left(\frac{z}{z_+}\right)^d$$

$z = 1/r$

$z_+ = 1/r_+$

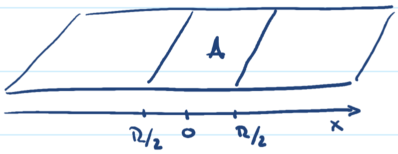
(not quite FG coordinates since these require $g_{zz} = \frac{1}{z^2}$)

The temperature is $T_H = \frac{1}{4\pi}\left|f'(z_+)\right| = \frac{d}{4\pi z_+}$

We could consider a strip or a ball. Say a strip

$\vec{x}_{d-1} = (x, \vec{x}_{d-2})$

$-R/2 \leq x \leq R/2$



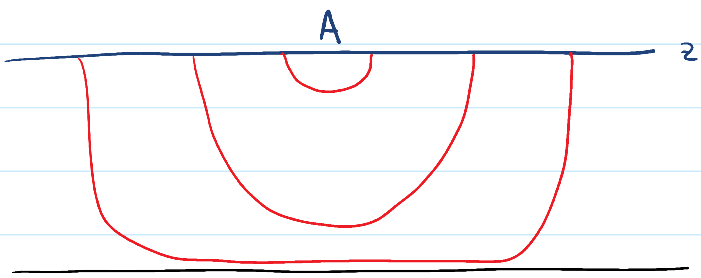
and parametrize $z(x)$. Then $\quad \mathrm{Vol}(\vec{x}_{d-2}) = V_{d-2}$

$$S(A) = \frac{1}{4G} V_{d-2} \int dx \, \frac{1}{z(x)^{d-1}} \sqrt{1 + \frac{z'(x)^2}{f(z(x))}}$$

One can then derive the equations, but they must be solved numerically (not too difficult).

The results are easy to interpret

Consider regions A of growing size R



$z = 0$

$z = z_+$: horizon (minimal surface)

For $R \ll z_+ \sim \frac{1}{T_H}$, $S(A)$ is almost like in vacuum

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$$S(A) \sim \frac{V_{d-2}}{\varepsilon^{d-2}} = \frac{\mathrm{Area}(\partial A)}{\varepsilon^{d-2}}$$

For $R \gg z_+ \sim \frac{1}{T_H}$, $S(A)$ gets an extensive volume contribution from the horizon

$$\sim R V_{d-2} T_H^{d-1} = \mathrm{Vol}(A)\, T_H^{d-1}$$

(The divergent part $\sim \frac{1}{\varepsilon^{d-2}}$ can be subtracted as a vacuum contribution, to leave a finite thermal entropy)